

On dualities for non-Abelian gauge theories with continuous center

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ABSTRACT: Formulating gauge theories for gauge groups admitting a continuous center can require to include charged scalars to define a gauge-coupling function. We show that the gauge-fields in the center can be dualized into form-fields of dimension-dependent degree. The resulting theory admits a smaller gauge group that factorizes out the center, but contains a Chern-Simons type term coupling the scalars to the form-fields. As an explicit example we consider the gauge group being the Heisenberg group and show that the dual action only admits an Abelian gauge symmetry. We comment on the vacuum symmetries in these settings, their supersymmetrization, and point out their importance in string theory.

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1 Introduction

A classical duality generalizing four-dimensional electro-magnetic duality is provided by the fact that in d -dimensions a massless Abelian vector admits a dual description in terms of a massless $(d - 3)$ -form. In order that such a dual theory exists it is crucial that the vector field only appears with additional derivatives in the action. Clearly, this is generally not true for non-Abelian gauge theories, since the bare gauge-fields can appear in the definition of the field strength. For certain classes of non-Abelian groups, however, a subset of the gauge-fields only occurs together with derivatives. Such groups are exactly the ones with continuous center, since by definition all gauge fields parameterizing the center commute with all other group elements. In this work we focus on such groups and study the associated gauge theory. We will also show that in this cases the duality to a description with $(d - 3)$ -forms can be performed. These dual descriptions are often crucial when studying effective actions of string theory.

Already the Lagrangian formulation of gauge theories with the gauge group admitting a continuous center is more involved than for standard Yang-Mills theories. This can be traced back to the fact that their Killing form has non-maximal rank and a simple kinetic term for the gauge-fields cannot be defined using the Killing form alone. To nevertheless define a kinetic term one is required to include extra scalar degrees of freedom that transform appropriately under the gauge group. One can then naturally introduce a field-dependent gauge-coupling function that is positive definite and non-vanishing along certain parts of the scalar field space. As expected this kinetic term will only depend on the derivative of gauge-fields along the center of the group, while the bare gauge fields from the center do not appear. This suggests that there indeed exists a dual description involving $(d - 3)$ -forms. As we will discuss in detail in this work this duality can be more involved due to required presence of the charged scalars. Treating the gauge fields in the center similarly to massive fields we find that nevertheless a dual description can be established. Remarkably, this dual formulation will only admit a smaller gauge group supplemented by Abelian local symmetries for the form fields.

This work can be motivated from various directions. Firstly, one realizes that the Heisenberg group is a key example for a group with continuous center. Generalizations of this group appear often as symmetry groups of string theory moduli spaces. Therefore one might wonder how these symmetry groups can be consistently gauged and how such gaugings can be detected in the effective action. Effective actions with gauged Heisenberg groups arise, for example, in compactifications of Type II string theory or M-theory on five- or six-dimensional manifolds with $SU(2)$ structure [1–6], or M-theory

on manifolds with $SU(4)$ structure [7]. Other reductions with non-Abelian gaugings of this type are reviewed, for example, in [8]. In such reductions the effective action might not immediately be in the correct frame to infer the full non-Abelian gauge group and several dualization steps have to be performed. Our work explains in detail what happens to the gauge groups in such dualizations and highlights the occurring curiosities. For example, while in the vector formulation with Heisenberg group one is dealing with a non-Abelian gauge theory, the dual theory with $(d - 3)$ -forms might only admit an Abelian gauge symmetry.

A second motivation is provided when studying the vacuum configurations in gauge theories with continuous center. In fact, due to the required presence of gauged scalars to define the theory, their vacua can have interesting discrete Abelian and non-Abelian gauge symmetries. Recall that discrete Abelian gauge symmetries such as \mathbb{Z}_p can be understood as arising by starting with a $U(1)$ gauge group and gauging a Higgs scalar in a non-linear fashion. In a dual description one can replace the $U(1)$ -gauge field by a $(d - 3)$ -form and the Higgs scalar by a $(d - 2)$ -form. This dual theory admits an emergent $U(1)$ gauge symmetry (see [9] for an in-depth discussion on such discrete Abelian symmetries). The non-Abelian generalization of this discussion arises naturally for gauge groups with continuous center. For example, starting with the continuous Heisenberg group the required coupled Higgs scalars can break it to the discrete Heisenberg group $H_{\mathbb{Z}}$ in the vacuum. Replacing the gauge fields of the center with dual $(d - 3)$ -forms and the coupling Higgs scalars by $(d - 2)$ -forms the resulting theory has an emergent symmetry group just as in the Abelian case. In contrast to the Abelian case, the emergent symmetry group generally differs from the original non-Abelian group.

The paper is organized as follows. In section 2 we discuss the gauge theory for the three-dimensional Heisenberg group. We argue that additional scalars are required to define a kinetic term, which at the same time allow for the vacuum symmetry group to be a discrete non-Abelian subgroup of H . The dualization of the vector spanning the center of H is performed in detail. The generalization of this construction to other groups with continuous center can be found in section 3. Again we first construct the non-Abelian gauge theory and then perform the dualization of the vectors parameterizing the center of the gauge group. We conclude in section 4 by commenting on the significance of the discussed gauge groups in string theory and provide a brief discussion on the supersymmetrization of the constructed actions.

2 Dual Heisenberg actions and discrete symmetries

In this section we introduce gauge theories that admit the three-dimensional Heisenberg group H as gauge group. This first simple example will highlight many key features of how to formulate a gauge theory for groups with continuous center. The Heisenberg gauge theory itself is introduced in subsection 2.1, where we will also comment on the discrete remnants of this group when considering vacuum configurations. In subsection 2.2 we perform the dualization of the vector parameterizing the center of H into a $(d - 3)$ -form. We discuss the symmetries of the dual action and show that it admits only Abelian gauge symmetries.

2.1 Heisenberg gauge theory and non-Abelian discrete symmetries

We first construct the action describing a gauge theory with the Heisenberg group H as gauge group in d space-time dimensions. We consider the simplest case in which H is three-dimensional. It takes the form

$$H = \mathbb{R} \ltimes (\mathcal{H}_1 \times \mathcal{H}_2), \quad (2.1)$$

where \mathcal{H}_i is either \mathbb{R} or $U(1)$. H is generated by three algebra elements t_A with $A = 1, 2, 3$ satisfying

$$[t_1, t_2] = -Mk t_3, \quad (2.2)$$

with all other commutators vanishing. The only non-trivial structure constants are $f_{ab}^3 = -Mk\epsilon_{ab}$, where ϵ_{ab} is the two-dimensional Levi-Civita symbol and $a, b = 1, 2$.

Next we introduce the gauge fields A^A for the generators t_A that transform under the gauge symmetry as

$$\begin{aligned} \delta A^a &= d\lambda^a, \\ \delta A^3 &= d\lambda^3 + Mk\epsilon_{ab}A^a\lambda^b - \frac{1}{2}Mk\epsilon_{ab}\lambda^a d\lambda^b. \end{aligned} \quad (2.3)$$

Note that this is the transformation law for actual finite group actions. At infinitesimal level only the first two terms in δA^3 are considered. The field strengths of A^A are denoted by $F^A = dA^A - \frac{1}{2}f_{BC}^A A^B \wedge A^C$. One readily checks that gauge invariance of their kinetic term

$$S_{\text{kin}}^{(d)} = - \int Q_{AB} F^A \wedge *F^B \quad (2.4)$$

implies that Q_{AB} has to transform non-trivially under H . More precisely, one has

$$Q_{AB} \rightarrow Q_{CD}(D^{-1})_A^C (D^{-1})_B^D, \quad (2.5)$$

where D_A^B is the adjoint of the group. For the Heisenberg group (2.3) one readily computes

$$(D_A^B) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ Mk\lambda^2 - Mk\lambda^1 & 1 & \end{pmatrix}, \quad (2.6)$$

which is independent of λ^3 . Note that the Heisenberg group has no positive-definite Killing form. To nevertheless find a positive-definite Q_{AB} one can introduce scalar fields charged under H and make Q_{AB} field-dependent.

There are two ways that this can be achieved which we discuss in the following. The minimal approach is to introduce two periodic scalars b^a that transform under (2.3) as

$$\delta b^a = k\lambda^a, \quad b^a \cong b^a + 1, \quad (2.7)$$

and admit gauge invariant derivatives

$$Db^a = db^a - kA^a. \quad (2.8)$$

The periodicity ensures that we find interesting vacuum configurations and allows us to connect with concrete string theory realizations. A gauge-coupling function Q_{AB} rendering (2.4) invariant then takes the form

$$(Q_{AB}) = \begin{pmatrix} \mathcal{M}_{ab} + (M)^2 \epsilon_{ac} \epsilon_{bd} b^c b^d & M \epsilon_{ca} b^c \\ M \epsilon_{cb} b^c & \mathcal{M} \end{pmatrix}, \quad (2.9)$$

where \mathcal{M}_{ab} is positive definite and \mathcal{M} is a positive constant. With these definitions at hand a gauge-invariant action is given by

$$S_{\min}^{(d)} = - \int Q_{AB} F^A \wedge *F^B + \mathcal{N}_{ab} Db^a \wedge *Db^b, \quad (2.10)$$

where \mathcal{N}_{ab} is positive definite and independent of b^a .

The space of inequivalent vacua of (2.10) is obtained for $A^A = 0$ and constant scalars $\langle b^a \rangle$. Constant gauge transformations preserving the $\langle b^a \rangle$ correspond to a preserved vacuum symmetry. The periodicity of the b^a in (2.7) yields to a breaking of the continuous Heisenberg symmetry H to

$$G_1 = \mathbb{Z}_k \ltimes (\mathcal{H}_1 \times \mathbb{Z}_k). \quad (2.11)$$

Note that this is a non-compact group for $\mathcal{H}_1 = \mathbb{R}$ and as such not expected to arise in a theory of quantum gravity. In contrast G_1 is a compact group for $\mathcal{H}_1 = U(1)$.

One might wonder if (2.10) can be extended such that the scalars break the symmetry to the discrete Heisenberg group $H_{\mathbb{Z}}$. In order to do that one has to introduce a third periodic scalar b^3 that transforms under the group H as

$$\delta b^3 = p\lambda^3 + \frac{Mp}{2}\epsilon_{ab}b^a\lambda^b, \quad b^3 \cong b^3 + 1, \quad (2.12)$$

with the transformations of the b^i in (2.7) unchanged. The covariant derivatives b^3 is given by

$$Db^3 = db^3 - pA^3 - \frac{Mp}{2}\epsilon_{ab}b^aA^b. \quad (2.13)$$

The full gauge-invariant action now takes the form

$$S_{\text{kin}}^{(d)} = - \int Q_{AB}F^A \wedge *F^B + \tilde{\mathcal{T}}_{AB}Db^A \wedge *Db^B, \quad (2.14)$$

where

$$(\tilde{\mathcal{T}}_{AB}) = \begin{pmatrix} \mathcal{N}_{ab} + \frac{(Mp)^2}{4k^2\mathcal{G}}\epsilon_{ac}\epsilon_{bd}b^cb^d & \frac{Mp}{2k\mathcal{G}}\epsilon_{ca}b^c \\ \frac{Mp}{2k\mathcal{G}}\epsilon_{cb}b^c & \frac{1}{\mathcal{G}} \end{pmatrix}. \quad (2.15)$$

Here \mathcal{G} is a positive constant.

The symmetry group of the vacua of (2.14) is now the discrete Heisenberg group $H_{\mathbb{Z}}$,

$$H_{\mathbb{Z}} = \mathbb{Z}_k \ltimes (\mathbb{Z}_p \times \mathbb{Z}_k). \quad (2.16)$$

Note that this group is compact. The appearance of $H_{\mathbb{Z}}$ in string theory is therefore in accordance with the ‘folk theorem’ that forbids non-compact gauge symmetries in a theory of quantum gravity.

To close this subsection we note that the states that can directly couple to the fields in (2.14) are simply particles. They can be charged under (the discrete remnants) of the Heisenberg group H . However, we know from general arguments about discrete gauge symmetries [10] that there are also charged $(d-1)$ -branes that couple minimally to the dual $(d-2)$ -forms of the scalars b^a .

2.2 Dualizing the action

In the next step we aim to study a dual frame describing the theories (2.10) and (2.14). This will yield us to discover emergent new gauge symmetries and a gauge group G_e that generally differs from all groups encountered so far.

Our starting point is the observation that the vector A^3 only appears with derivatives in the action (2.10). This implies that it should be possible to replace it by a dual

degree of freedom which is given by a $(d-3)$ -form B . This dualization proceeds in the standard way. Firstly, we replace dA^3 with \tilde{F}^3 everywhere in (2.14). Then we add a Lagrange multiplier term to arrive at the parent action

$$S_{\text{par}}^{(d)} = S_{\text{min}}^{(d)}(dA^3 \rightarrow \tilde{F}^3) + 2 \int dB \wedge \tilde{F}^3. \quad (2.17)$$

The equations of motion for B simply imply $d\tilde{F}^3 = 0$, such that locally one can write $\tilde{F}^3 = dA^3$. Inserted back into (2.17) we arrive back at the original action (2.10). However, we can also choose to eliminate \tilde{F}_3 and keep B . Before doing this, let us comment on the symmetries of (2.17). The fundamental fields in (2.17) are A^i , b^i and B , \tilde{F}^3 . The symmetry transformations for \tilde{F}^3 are identical to those of dA^3 and inferred from (2.3). They capture the non-Abelian structure of H and are a symmetry of (2.17) up to a total derivative. Note that in contrast B has only an Abelian symmetry. After eliminating \tilde{F}^3 from (2.17) the dual action reads

$$S_{\text{min-e}}^{(d)} = - \int \mathcal{M}_{ab} dA^a \wedge *dA^b + \mathcal{N}_{ab} Db^a \wedge *Db^b + \mathcal{M}^{-1} dB \wedge *dB \\ + \frac{M}{k} \epsilon_{ab} dB \wedge Db^a \wedge Db^b. \quad (2.18)$$

Note that seemingly the non-Abelian structure of H has disappeared, since all the covariant derivatives are Abelian and an Abelian symmetry of B has emerged. Furthermore, (2.18) contains a particular new Chern-Simons type term. The seemingly higher-derivative part of this term is simply a total derivative and the chosen form allowed us to highlight its gauge invariance. As we will see in the remaining parts of this work, precisely terms of this type signal the presence of a non-Abelian gauge group in the dual description.

Let us now turn to the discussion of the second action (2.14) that admits the discrete Heisenberg group $H_{\mathbb{Z}}$ as vacuum symmetry. In this case also A^3 is appearing without derivatives in the covariant derivative (2.13) of the scalar b^3 . This implies that A^3 is massive and has to be dualized into a massive form field. In other words, a general dualization of A^3 into B can now only be performed when dualizing b^3 at the same time into a $(d-2)$ -form V_{d-2} . The parent action for (2.14) now takes the form

$$S_{\text{par}}^{(d)} = - \int \tilde{\mathcal{T}}_{AB} Db^A \wedge *Db^B + \mathcal{M}_{ab} dA^a \wedge *dA^b \\ + \mathcal{M}^{-1} DB \wedge *DB + 2DB \wedge U, \quad (2.19)$$

where

$$DB = dB - pV \\ U = dA^3 + \frac{Mk}{2} \epsilon_{ab} A^a \wedge A^b + M \epsilon_{ab} b^a dA^b. \quad (2.20)$$

It is interesting to notice that this parent action has the local symmetries given by (2.3), (2.7), and (2.12) supplemented by

$$\delta V = d\Lambda, \quad \delta B = p\Lambda, \quad (2.21)$$

where Λ is a $(d-3)$ -form. This corresponds to the group $H \times U(1)_{d-3}$.¹ Notice that the transformation rules given above are valid for finite gauge parameters λ and that U is gauge-invariant. The parent action (2.19) is build to precisely yield the action (2.14) upon using the equations of motion of V and B to eliminate these fields from the theory.

Alternatively we can use the equations of motion of b^3 , A^3 and eliminate these degrees of freedom from the action. The result is the theory

$$\begin{aligned} S_e^{(d)} = & - \int \mathcal{N}_{ab} D b^a \wedge * D b^b + \mathcal{M}_{ab} dA^a \wedge * dA^b + \mathcal{G} dV \wedge * dV \\ & + \mathcal{M}^{-1} DB \wedge * DB + \frac{M}{k} \epsilon_{ab} DB \wedge D b^a \wedge D b^b \end{aligned} \quad (2.22)$$

This action has the Abelian gauge symmetry $U(1)^2 \times U(1)_{d-3}$ that acts on the different fields as

$$\delta A^a = d\lambda^a, \quad \delta b^a = k\lambda^a, \quad \delta V_{d-2} = d\Lambda, \quad \delta B = p\Lambda, \quad (2.23)$$

for arbitrary real functions λ^a and $(d-3)$ -form Λ .

3 Dual actions for gauge theories with continuous center

In this section we provide the generalisation of the duality introduced in section 2 to other gauge groups with continuous center. In subsection 3.1, we discuss properties of the gauge groups of interest and stepwise introduce a d -dimensional gauge-invariant action. This requires to include scalar fields that transform under such groups. In subsection 3.2 we propose the duality that replaces the gauge fields parameterizing the center with $(d-3)$ -forms. As for the Heisenberg group discussed in section 2 the dual theory has a simpler gauge group but admits additional topological couplings of Chern-Simons type.

¹The subscript in $U(1)_{d-3}$ is included to stress that the corresponding gauge parameter is a $(d-3)$ -form.

3.1 Gauge theories for groups with continuous center

To begin with, we like to introduce the gauge groups to which a duality similar to the one of section 2 can be applied. We denote the considered Lie group by G and name its Lie algebra \mathfrak{g} . The Lie algebra generators $\{t_A\}$ satisfy

$$[t_A, t_B] = f_{AB}^C t_C, \quad (3.1)$$

with f_{AB}^C being the structure constants of \mathfrak{g} . We want to focus on a particular class of Lie algebras, with the following property. Split the generators into $\{t_a, t_\alpha\}$ with $a = 1, \dots, r$ and $\alpha = 1, \dots, n$ and assume they satisfy that

$$f_{\alpha B}^A = 0. \quad (3.2)$$

This condition states that the subspace spanned by $\{t_\alpha\}$ is an Abelian ideal \mathfrak{h} of \mathfrak{g} . We consider the maximal subset of generators $\{t_\alpha\}$ for which (3.2) can be satisfied, i.e. the case where \mathfrak{h} is the center of \mathfrak{g} and n is its dimension. Note that \mathfrak{h} corresponds to the algebra of $U(1)^n$. The quotient $\tilde{\mathfrak{g}} = \mathfrak{g}/\mathfrak{h}$ is a Lie algebra spanned by $\{t_a\}$ with structure constants f_{ab}^c . One can easily check that if f_{AB}^C , constrained by (3.2), satisfies the Jacobi identities, then f_{ab}^c does too. In more fancy terms, \mathfrak{g} is a central extension of $\tilde{\mathfrak{g}}$ by \mathfrak{h} . We denote the Lie group associated to $\tilde{\mathfrak{g}}$ by \tilde{G} .

An important feature of \mathfrak{g} is that its Killing form $B_{AB} = f_{AC}^D f_{BD}^C$ is degenerate. Using (3.2) one finds that

$$B_{AB} = \begin{pmatrix} f_{ac}^d f_{bd}^c & 0 \\ 0 & 0 \end{pmatrix}, \quad (3.3)$$

where the upper non-vanishing entries correspond to the Killing form of $\tilde{\mathfrak{g}}$. The form (3.3) implies that B_{AB} cannot be simply used as the gauge coupling function for a gauge theory with gauge group G . Similar to the discussion in section 2.1 an action for a gauge theory with group G has to be more involved. We discuss its construction next.

To build a gauge theory for the group G we first introduce a collection of gauge fields A^A with field strength $F^A = dA^A - \frac{1}{2}f_{BC}^A A^B \wedge A^C$. Setting $F = F^B t_B$ and $A = A^B t_B$, a gauge transformation $U \in G$ can be represented as

$$A \rightarrow UAU^{-1} - U dU^{-1}, \quad F \rightarrow U F U^{-1}. \quad (3.4)$$

If $U = e^{\lambda^A t_A}$ we have, for small λ^A ,

$$\delta A^A = d\lambda^A - f_{BC}^A A^B \lambda^C, \quad \delta F^A = -f_{BC}^A F^B \lambda^C. \quad (3.5)$$

Note that F^A transforms in the adjoint of the group G and we will denote this representation in the previous section by $D[U]_A^B$ for $U \in G$. Explicitly, we have

$$D[U]_A^B = e^{\lambda^C f_{CA}^B}, \quad (3.6)$$

where $\lambda^C f_{CA}^B$ has to be understood as a matrix in the indices A, B that gets exponentiated.

Next we want to introduce the action of the d -dimensional gauge theory. Consider the definition

$$S^{(d)} = - \int Q_{AB} F^A \wedge *F^B + \tilde{S} \quad (3.7)$$

where Q_{AB} is a symmetric real matrix and \tilde{S} is gauge invariant by itself. Gauge invariance of the first term implies that Q_{AB} must transform as

$$Q_{AB} \rightarrow Q_{CD} D^{-1}[U]_A^C D^{-1}[U]_B^D, \quad (3.8)$$

as already discussed in (2.5). The Killing form B_{AB} transforms precisely as (3.8), but is degenerate for the considered G . Therefore, Q_{AB} must depend on extra scalar fields transforming under G .

Consider adding to the theory a set of scalars $\{\phi^A\} = \{\phi^a, \phi^\alpha\}$ living in the group G . More precisely, we have that $V(\phi) \equiv e^{\phi^A t_A} \in G$. Since G is a Lie group, we can always introduce a Riemannian metric on it such that the group of isometries is (at least) G itself. We declare that these transform under the gauge group G as

$$V(\phi) \rightarrow V(\phi) e^{-\Theta \lambda}, \quad (3.9)$$

with $\Theta \lambda \equiv \Theta_B^A \lambda^A t_B$ for some constant Θ_B^A that encodes the particular gauging we are introducing. One typically does that by considering the right-invariant forms

$$\tilde{\eta} = dV(\phi) V^{-1}(\phi) \quad (3.10)$$

which are elements in \mathfrak{g} invariant under (3.9) provided λ is independent of the spacetime coordinates. Since we are interested in local symmetries, we modify the above as

$$\eta = dV(\phi) V^{-1}(\phi) + V(\phi) \Theta A V^{-1}(\phi) \equiv \eta^A t_A. \quad (3.11)$$

where $\Theta A \equiv \Theta_B^A A^B t_A$. This η is invariant under (3.4) and (3.9) with $\lambda = \lambda(x)$ if Θ satisfies that

$$f_{AB}^D \Theta_D^C = f_{EF}^C \Theta_A^E \Theta_B^F, \quad (3.12)$$

which can be recognized as the quadratic constraint. It ensures that the scalars ϕ^A are charged under a subgroup of G . In order to dualize the action, we need to make sure that η^a does not depend on A^α which amounts to imposing

$$\Theta_\alpha^b = 0. \quad (3.13)$$

Then, eq.(3.12) projects to the quadratic constraint on $\tilde{\mathfrak{g}}$ as well as

$$f_{ab}^c \Theta_c^\alpha + f_{ab}^\beta \Theta_\beta^\alpha = f_{ef}^\alpha \Theta_a^e \Theta_b^f. \quad (3.14)$$

This means that if the extension is non-trivial ($f_{ab}^\alpha \neq 0$) and a subgroup of \tilde{G} is gauged ($\Theta_a^b \neq 0$), then we necessarily need to have a gauging along the centre ($\Theta_\beta^\alpha \neq 0$ or $\Theta_b^\alpha \neq 0$).

The extra term in the action is therefore

$$\tilde{S} = - \int \mathcal{T}_{AB} \eta^A \wedge * \eta^B, \quad (3.15)$$

which is gauge invariant and contains the kinetic term for the scalars ϕ^A . Here \mathcal{T}_{AB} is a scalar product in the Lie algebra \mathfrak{g} so it is independent of ϕ . Expanding (3.15) in terms of the fields ϕ^A one finds the Riemannian metric mentioned above. However, it is better to leave it the way it is since it makes gauge invariance manifest. Using the properties of G as well as (3.13), we find that

$$\eta^\alpha = d\phi^\alpha + \frac{1}{2} f_{bc}^\alpha q^{bc} + \Theta_\beta^\alpha A^\beta + (\Theta_a^\alpha + D[V]_b^\alpha \Theta_a^b) A^a. \quad (3.16)$$

where q^{bc} is such that $f_{bc}^\alpha dq^{bc} = f_{bc}^\alpha \tilde{\eta}^b \wedge \tilde{\eta}^c$, due to the Maurer-Cartan structure equations.

We are now ready to build the gauge coupling function Q_{AB} using the scalars $V(\phi)$. If we choose

$$Q_{AB} = \mathcal{S}_{CD} D[V]_E^C \Theta_A^E D[V]_F^D \Theta_B^F \quad (3.17)$$

with \mathcal{S}_{AB} a real non-degenerate symmetric matrix independent of $V(\phi)$, then the kinetic term for the bosons is gauge invariant. Under a gauge transformation (3.9) we have that

$$D[V]_A^B \rightarrow D[V e^{-\Theta\lambda}]_A^B = D[V]_A^C D[e^{-\Theta\lambda}]_C^B \quad (3.18)$$

such that Q_{AB} transforms as desired in (3.8).²

²Using the quadratic constraint (3.12) we find that $D[e^{-\Theta\lambda}]_A^B \Theta_C^A = \Theta_A^B D[e^{-\lambda}]_C^A$.

Note that the adjoint representation for the considered groups G satisfies various special properties. Using (3.2) one readily shows that it takes the form

$$\begin{aligned} D[U]_a^b &= \delta_a^b + f_{cd}^b \lambda^c \tilde{D}[U]_a^d, & D[U]_\alpha^b &= 0, \\ D[U]_a^\beta &= f_{cd}^\beta \lambda^c \tilde{D}[U]_a^d, & D[U]_\alpha^\beta &= \delta_\alpha^\beta, \end{aligned} \quad (3.19)$$

where $D[U]_a^b$ is precisely the adjoint representation of \tilde{G} and $\tilde{D}[U]_a^b$ is a matrix that only depends on f_{ab}^c and can be computed explicitly. Notice that $D[U]_A^B$ does not depend on λ^α since t_α generate the centre of G . Using the properties (3.19) we find that also Q_{AB} has a special form,

$$Q_{ab} = \mathcal{S}_{cd} D[V]_e^c \Theta_a^e D[V]_f^d \Theta_b^f + \mathcal{S}_{\alpha\beta} (D[V]_c^\alpha \Theta_a^c + \Theta_a^\alpha) (D[V]_d^\beta \Theta_b^\beta + \Theta_b^\beta), \quad (3.20)$$

$$Q_{a\beta} = \mathcal{S}_{\alpha\gamma} (D[V]_b^\alpha \Theta_a^\beta + \Theta_a^\alpha) \Theta_\beta^\gamma \quad Q_{\alpha\beta} = \mathcal{S}_{\gamma\rho} \Theta_\alpha^\gamma \Theta_\beta^\rho. \quad (3.21)$$

where we set $\mathcal{S}_{ab} = 0$ for simplicity.

To summarize, the full action corresponding to the gauge theory for G coupled to the scalars ϕ^A is

$$S^{(d)} = - \int Q_{AB} F^A \wedge *F^B + \mathcal{T}_{AB} \eta^A \wedge *\eta^B, \quad (3.22)$$

which is invariant under the gauge transformations (3.4) and (3.9) for every $U \in G$. It is useful to split the index range which allows to rewrite the action as

$$S^{(d)} = - \int \tilde{Q}_{ab} F^a \wedge *F^b + \mathcal{S}_{\alpha\beta} M^\alpha \wedge *M^\beta + \mathcal{T}_{AB} \eta^A \wedge *\eta^B \quad (3.23)$$

where we have defined

$$\tilde{Q}_{ab} = \mathcal{S}_{cd} D[V]_e^c \Theta_a^e D[V]_f^d \Theta_b^f, \quad M^\alpha = \Theta_\beta^\alpha F^\beta + (D[V]_c^\alpha \Theta_a^c + \Theta_a^\alpha) F^a. \quad (3.24)$$

The expression (3.23) is useful because it splits the kinetic term for the gauge fields in G in two terms that are gauge invariant independently. The first is simply the kinetic term for the gauge fields in $\tilde{G} = G/U(1)^n$ while the second term is a combination of A^α and A^a that is gauge invariant by itself since M^α is invariant. In particular, it depends on the structure constants f_{ab}^α that encode the information about the extension of $\tilde{\mathfrak{g}}$ by \mathfrak{h} .

3.2 Dualization of the action

Having introduced in detail the d -dimensional gauge theories with gauge groups G , we are now in the position to perform a dualization similar to the one of section 2.2.

Since we introduced a full representation of scalars ϕ^A in (3.22) we need to perform the analogue steps that yielded the actions (2.19) and (2.22).

When performing the dualization we first note that the precise way to do that depends on the rank of Θ_α^β . As seen in (3.16) this tensor dictates how many scalars ϕ^α are shift-gauged by the A^α that we like to dualize. As we have seen in section 2.2 the dualization slightly differs for shift-gauging and un-gauged vectors. Therefore, in order to present the parent action, we will assume that Θ_α^β has maximal rank. The dual action, however, is independent of the rank of Θ_α^β and holds generally. We thus dualize ϕ^α into $(d-2)$ -forms V_α and A^α into $(d-3)$ -forms B_α . In order to do that we propose the following parent action

$$S_{\text{par}}^{(d)} = - \int \tilde{Q}_{ab} F^a \wedge *F^b + \mathcal{T}_{AB} \eta^A \wedge *\eta^B + \mathcal{S}^{-1\alpha\beta} DB_\alpha \wedge *DB_\beta + 2DB_\alpha \wedge M^\alpha \quad (3.25)$$

where we defined

$$DB_\alpha = dB_\alpha + \Theta_\alpha^\beta V_\beta. \quad (3.26)$$

The independent variables are ϕ^A, A^A, B_α , and V_α . This parent action is invariant under $G \times U(1)^n$ given by (3.4) and (3.9) together with

$$B_\alpha \rightarrow B_\alpha - \Theta_\alpha^\beta \lambda_\beta, \quad V_\alpha \rightarrow V_\alpha + d\lambda_\alpha \quad (3.27)$$

where λ_α is an arbitrary $(d-3)$ -form.

On the one hand, computing the equation of motion for V_α and inserting the result into (3.25) we obtain the original action (A.1). On the other hand, when we use the equations of motion for A^α we arrive at the dual action

$$S_e^{(d)} = - \int \tilde{Q}_{ab} F^a \wedge *F^b + \mathcal{T}_{ab} \eta^a \wedge *\eta^b + \mathcal{S}^{-1\alpha\beta} DB_\alpha \wedge *DB_\beta + \mathcal{T}^{-1\alpha\beta} \Theta_\beta^\rho \Theta_\alpha^\gamma dV_\rho \wedge *dV_\gamma - f_{ab}^\alpha DB_\alpha \wedge \eta^a \wedge \eta^b. \quad (3.28)$$

where we have assumed that $\mathcal{T}_{a\beta} = 0$ to make the computations simpler. This Lagrangian is invariant under $\tilde{G} \times U(1)_{d-3}^n$.

This result is a generalization of the actions found in section 2.2 for the Heisenberg group H . To make this match precise, note that in the case of the Heisenberg group we have that $f_{ab}^c = 0$ and $f_{ab}^3 = \mu\epsilon_{ab}$ so the adjoint representation reads

$$D[V]_a^b = \delta_a^b, \quad D[V]_a^3 = \mu\epsilon_{ca}\phi^c. \quad (3.29)$$

Thus, the gauged right-invariant forms are

$$\begin{aligned}\eta^a &= d\phi^a + \Theta_b^a A^b, \\ \eta^3 &= d\phi^3 + \Theta_3^3 A^3 + \frac{1}{2}\mu\epsilon_{ab}\phi^a d\phi^b + (\Theta_a^3 + \mu\epsilon_{cb}\phi^c\Theta_a^b)A^a.\end{aligned}\tag{3.30}$$

Comparing with section 2 we find that $\mu = -Mk$ and that the gaugings are given by

$$\Theta_a^b = -k\delta_a^b, \quad \Theta_a^3 = 0, \quad \Theta_3^3 = k^2.\tag{3.31}$$

The fields and coupling functions are then identified as

$$\begin{aligned}\phi^a &= b^a, \quad \phi^3 = -\frac{k^2}{p}b^3, \quad V_3 = -pV, \quad B_3 = \frac{1}{k^2}B, \\ \mathcal{S}_{ab} &= k^{-2}\mathcal{M}_{ab}, \quad \mathcal{T}_{ab} = \mathcal{N}_{ab}, \quad \mathcal{S}_{33} = k^{-4}\mathcal{M}.\end{aligned}\tag{3.32}$$

4 Conclusions

In this paper we studied the gauge theories for gauge groups G admitting a continuous n -dimensional center $U(1)^n$. We introduced the kinetic terms for the gauge fields by appropriately coupling the theory to scalar fields charged under G . These scalars ensure the existence of a positive definite gauge-coupling function and allow the settings to have an interesting vacuum structure. We have exemplified this fact by gauging the three-dimensional Heisenberg group H and studied its breaking to its discrete version $H_{\mathbb{Z}}$ in the vacuum. Furthermore, we have shown that the gauge fields in the center of G can be dualized to $(d-3)$ -forms even in the case that G is non-Abelian. In order to perform this duality one generally has to also dualize some of the scalar degrees of freedom into $(d-2)$ -forms if they were non-trivially charged under the center gauge fields. The resulting dual theory has a smaller vector gauge group $\tilde{G} = G/U(1)^n$, supplemented by a $U(1)^n$ form-field gauge group. Interestingly, for the Heisenberg group H the dual group $\tilde{H} = U(1) \times U(1)$ is an Abelian group. However, the original non-Abelian structure is not lost, but reappears in a Chern-Simons like term that depends on the structure constants f_{ab}^α for the center and non-center elements.

Let us note that the structures we find in this work are expected to appear rather universally in string theory. To see this, one has to recall that the field strengths are typically involving a coupling to lower-degree forms. For example, in Type II string theories one has $F_{p+1} = dC_p + C_{p-3} \wedge H_3$, where H_3 is the NS-NS three-form field strength. By duality the degrees of freedom in C_p are mapped to the degrees of freedom in C_{8-p} . When working with the lower-degree forms the modification of

the field strength will be absent, which can hide the non-Abelian structure in effective theories upon dimensional reduction. In other words the effective theory is the one with $(d-3)$ -forms and Chern-Simons terms that we found after dualization. Examples of this feature can be found, e.g. in [4–7, 12]. The physics then only becomes fully transparent at the level of the Lagrangian when dualized to the vector formulation. This is equally true when coupling space-time filling D-branes to the setting as was recently discussed in [7].

As an interesting generalization one can consider the coupling of additional matter charged under the gauge group G . Such matter can be straightforwardly coupled in the vector formulation of the theory. The required additional couplings will depend on the bare gauge-fields of the center and it seems that a dual formulation with $(d-3)$ -forms does in general no longer exist. In contrast, when recalling the implementation in M-theory compactifications to three dimensions and their lift to F-theory [7], one can be tempted to think that such a dual description still exists, at least in three spacetime dimensions. It would be very interesting to explore this possibility further and clarify the application of this duality in F-theory compactifications with a charged matter spectrum. This is particular interesting due to the fact that often a non-Abelian discrete gauge symmetry remains as a selection symmetry. For example, the use of the discrete Heisenberg group as a selection symmetry has recently been discussed in [7, 12, 13]. Such selection rules can be of profound physical importance when studying allowed couplings in string theory effective actions.

It is also interesting to point out that the supersymmetrization of the actions we found in section 3.2 can be challenging. While the vector formulation seems to admit a rather straightforward supersymmetrization, this is not necessarily true for the dual $(d-3)$ -form formulation. By duality we expect that it exists and it would be desirable to give general supersymmetric forms for the action (3.28) directly. Clearly, the actions will depend on the space-time dimension and the number of supersymmetries one wants to realize. An interesting example are theories with $d=3$ with $\mathcal{N}=2$ supersymmetry for which preliminary results are recorded in appendix A.

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A Supersymmetrization for $d = 3$ with $\mathcal{N} = 2$

In this appendix we propose a way to make an Abelian gauging in $d = 3$ compatible with $\mathcal{N} = 2$ supersymmetry which is the completion of the Lagrangian (2.22) for $d = 3$. In three dimensions, the field B that appears in (2.22) is a scalar and V a vector. Thus, we work with a set of scalars f^i which include both the b^a and B as well as a set of vectors A^A including A^a and V . We discuss how to make the non-standard coupling $DB \wedge Db^a \wedge Db^b$ supersymmetric. In order to make supersymmetry manifest we work in $\mathcal{N} = 2$ superspace and follow the conventions introduced in appendix A of [11].

Let us start with a non-linear sigma model given by

$$S = \int d^2\theta d^2\bar{\theta} K(Q^i, \bar{Q}^i), \quad (\text{A.1})$$

where $K(Q^i, \bar{Q}^i)$ is the Kähler potential for the chiral superfields Q^i . In components these are

$$Q^i = f^i + \theta^\alpha \psi_\alpha^i + \theta^2 F^i + i\theta^a \bar{\theta}^\beta \partial_{\alpha\beta} f^i + \frac{i}{2} \theta^2 \bar{\theta}^\alpha \partial_{\alpha\beta} \psi^{i\beta} + \frac{1}{4} \theta^2 \bar{\theta}^2 \square f^i. \quad (\text{A.2})$$

with $\partial_{\alpha\beta} = (\gamma^m)_{\alpha\beta} \partial_m$. Let us assume that $K(Q^i, \bar{Q}^i)$ is such that its Lie derivative along the vectors $X_A = \Theta_A^i (\partial_i + \bar{\partial}_i)$ vanishes, namely,

$$L_{X_A} K = \Theta_A^i K_i + \Theta_A^i \bar{K}_{\bar{i}} = 0, \quad (\text{A.3})$$

where the quantities Θ_A^i are real constants. This implies that the Kähler metric for f^i has some Abelian isometries given by shifting the coordinates,

$$\delta_\lambda f^i = \Theta_A^i \lambda^A, \quad \delta_\lambda \bar{f}^i = \Theta_A^i \lambda^A, \quad (\text{A.4})$$

where λ^A are arbitrary real constants.

In order to gauge such isometries in a supersymmetric way, we need to introduce vector superfields V^A which have an expansion

$$V^A = \theta^\alpha \bar{\theta}^\beta A_{\alpha\beta}^A + i\theta^\alpha \bar{\theta}_\alpha \phi^A + i\theta^2 \bar{\theta}^\alpha \bar{\lambda}_\alpha^A - i\bar{\theta}^2 \theta^\alpha \lambda_\alpha^A + \theta^2 \bar{\theta}^2 D^A \quad (\text{A.5})$$

with $A_{\alpha\beta}^A = (\gamma^m)_{\alpha\beta} A_m^A$. The parameter λ^A gets replaced by a chiral superfield Λ^A under which V^A transforms as

$$\delta_\Lambda V^A = -\frac{i}{2} (\Lambda^A - \bar{\Lambda}^A), \quad (\text{A.6})$$

and the rule (3.2) becomes

$$\delta_\Lambda Q^i = \Theta_A^i \Lambda^A, \quad \delta_\Lambda \bar{Q}^i = \Theta_A^i \bar{\Lambda}^A \quad (\text{A.7})$$

The first step is to modify the Kähler potential to make it gauge invariant. This is achieved replacing (A.1) by

$$S_{kin} = \int d^2\theta d^2\bar{\theta} K(M^i, \bar{M}^i) \quad (\text{A.8})$$

where we defined

$$M^i = Q^i - i\Theta_A^i V^A, \quad \bar{M}^i = \bar{Q}^i + i\Theta_A^i V^A. \quad (\text{A.9})$$

These now transform as

$$\delta_\Lambda M^i = \delta_\Lambda \bar{M}^i = \Theta_A^i \text{Re } \Lambda^A, \quad (\text{A.10})$$

which ensures gauge invariance of (A.8). The bosonic term that arises from the above is (up to total derivatives)

$$\mathcal{L}_{kin} = -K_{i\bar{j}} D_m f^i D^m \bar{f}^j - K_{i\bar{j}} F^i \bar{F}^j - 2iK_i k_A^i D^A + K_{i\bar{j}} k_A^i k_B^j \phi^A \phi^B \quad (\text{A.11})$$

where the covariant derivative is $Df^i = df^i - \Theta_A^i \Lambda^A$, so only the real part of the scalar field is actually gauged.

As it stands, this is not an interesting theory because the equation of motion for D^A implies that $K_i \Theta_A^i = 0$ which, if plugged back into the action, gives the ungauged theory. Thus, we need to introduce extra terms that contain D^A to make it interesting. The usual solution is to introduce supersymmetric Chern-Simons terms

$$S_{CS} = \frac{i}{2} \int d^2\theta d^2\bar{\theta} \Theta_{AB} V^A \bar{D}^\alpha D_\alpha V^B \quad (\text{A.12})$$

with Θ_{AB} a symmetric constant matrix and D_α the covariant spinor derivatives (see appendix A of [11] for the conventions). The bosonic terms that this produces are

$$\mathcal{L}_{CS} = \frac{1}{2} \Theta_{AB} A^A \wedge F^B - 2\Theta_{AB} \phi^A D^B \quad (\text{A.13})$$

which contains a linear piece in D^A . One can also introduce kinetic terms for the vectors which produce quadratic terms in D^A .

Here we propose a different way to make the theory consistent which does not rely on CS or kinetic terms for the vectors by adding the following D-term couplings

$$S_{extra} = \int d^2\theta d^2\bar{\theta} \left(T_{ijk} \text{Im} M^i \nabla_\alpha Q^j \bar{\nabla}^\alpha \bar{Q}^k + \frac{i}{2} R_{Aij} \text{Im} M^i \text{Im} M^j \bar{D}^\alpha D_\alpha V^A \right). \quad (\text{A.14})$$

Let us try to understand these couplings. First of all, T_{ijk} is a constant tensor that needs to satisfy $T_{ijk} = -\bar{T}_{ikj}$ to make S_{extra} real and R_{Aij} is also constant and symmetric in ij . We also defined the covariant derivatives

$$\nabla_\alpha Q^i = D_\alpha(Q^i - 2i\Theta_A^i V^A), \quad \bar{\nabla}_\alpha \bar{Q}^i = \bar{D}_\alpha(\bar{Q}^i + 2i\Theta_A^i V^A) \quad (\text{A.15})$$

which have the nice property that

$$\delta_\Lambda(\nabla_\alpha Q^i) = \delta_\Lambda(\bar{\nabla}_\alpha \bar{Q}^i) = 0, \quad (\nabla_\alpha Q^i)^* = \bar{\nabla}_\alpha \bar{Q}^i. \quad (\text{A.16})$$

Then, due to (A.10) we have that $\delta_\Lambda \text{Im} M^i = 0$ which shows that (A.14) is gauge invariant. Although the first term is naively higher derivative, one can see by expanding (A.14) in components that it is not, just like the last term in (2.22). Finally, in order to being able to remove the auxiliary fields in a consistent way we need to impose that $\Pi_B^A R_{Aij} = R_{Bij}$ where $\Pi_B^A = \check{\Theta}_i^A \Theta_B^i$ with $\check{\Theta}_i^A$ the Moore-Penrose pseudoinverse.

The full expansion of (A.14) in components is rather long so we just point out that it contains the following couplings

$$\begin{aligned} & T_{ijk}(Df^i + D\bar{f}^i) \wedge Df^j \wedge D\bar{f}^k \\ & (R_{Aij} + \text{Im} T_{ijk} \Theta_A^i) \text{Im} f^i (Df^j + D\bar{f}^j) \wedge F^A \\ & \text{Re} T_{ijk} \Theta_A^i \text{Im} f^i (Df^j - D\bar{f}^j) \wedge F^A. \end{aligned} \quad (\text{A.17})$$

The first term has exactly the same structure as the coupling in (2.22), as desired. Now one can add the kinetic terms for the vectors in the standard way to complete the supersymmetrization of (2.22). However, we note that this is not necessary and the theory makes sense even if we do not include them.

References

- [1] D. Cassani, G. Dall’Agata and A. F. Faedo, “Type IIB supergravity on squashed Sasaki-Einstein manifolds,” *JHEP* **1005**, 094 (2010) [arXiv:1003.4283 [hep-th]].
- [2] J. T. Liu, P. Szepietowski and Z. Zhao, “Consistent massive truncations of IIB supergravity on Sasaki-Einstein manifolds,” *Phys. Rev. D* **81**, 124028 (2010) [arXiv:1003.5374 [hep-th]].
- [3] J. P. Gauntlett and O. Varela, “Universal Kaluza-Klein reductions of type IIB to N=4 supergravity in five dimensions,” *JHEP* **1006**, 081 (2010) [arXiv:1003.5642 [hep-th]].
- [4] T. Danckaert, J. Louis, D. Martinez-Pedrera, B. Spanjaard and H. Triendl, “The N=4 effective action of type IIA supergravity compactified on SU(2)-structure manifolds,” *JHEP* **1108**, 024 (2011) [arXiv:1104.5174 [hep-th]].

- [5] A. K. Kashani-Poor, R. Minasian and H. Triendl, “Enhanced supersymmetry from vanishing Euler number,” *JHEP* **1304**, 058 (2013) [arXiv:1301.5031 [hep-th]].
- [6] T. W. Grimm, A. Kapfer and S. Lust, “Partial Supergravity Breaking and the Effective Action of Consistent Truncations,” *JHEP* **1502**, 093 (2015) [arXiv:1409.0867 [hep-th]].
- [7] T. W. Grimm, T. G. Pugh and D. Regalado, “Non-Abelian discrete gauge symmetries in F-theory,” arXiv:1504.06272 [hep-th].
- [8] H. Samtleben, “Lectures on Gauged Supergravity and Flux Compactifications,” *Class. Quant. Grav.* **25**, 214002 (2008) [arXiv:0808.4076 [hep-th]].
- [9] T. Banks and N. Seiberg, “Symmetries and Strings in Field Theory and Gravity,” *Phys. Rev. D* **83**, 084019 (2011) [arXiv:1011.5120 [hep-th]].
- [10] K-M. Lee, “Non-Abelian discrete gauge theory,” Dissertation (Ph.D.), California Institute of Technology.
- [11] I. L. Buchbinder, N. G. Pletnev and I. B. Samsonov, “Effective action of three-dimensional extended supersymmetric matter on gauge superfield background,” [arXiv:1003.4806 [hep-th]].
- [12] M. Berasaluce-Gonzalez, P. G. Camara, F. Marchesano, D. Regalado and A. M. Uranga, “Non-Abelian discrete gauge symmetries in 4d string models,” *JHEP* **1209** (2012) 059 [arXiv:1206.2383 [hep-th]].
- [13] F. Marchesano, D. Regalado and L. Vazquez-Mercado, “Discrete flavor symmetries in D-brane models,” *JHEP* **1309**, 028 (2013) [arXiv:1306.1284 [hep-th]].